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BRIEF PAPER

BANACH FRAMES GENERATED BY COMPACT OPERATORS ASSOCIATED WITH A BOUNDARY VALUE PROBLEM

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Abstract. In this paper we give a type of a compact linear operator associated with a given boundary value problem which can generate a Banach frame for the underlying space.

Keywords: Hilbert space frame, Banach frame, boundary value problem, compact operator.

AMS Subject Classification: 42C15, 42C30, 42C05, 46B15; 35G45.

1. Introduction and motivation

In 1952, Duffin and Schaeffer, while addressing some deep problems in nonharmonic Fourier series define frames for Hilbert spaces [7]. Duffin and Schaeffer abstracted the fundamental notion of time-frequency atomic decomposition for signal processing by Gabor [11]. These ideas did not generate much interest outside of nonharmonic Fourier series and signal processing until the landmark paper of Daubechies, Grossmann and Meyer [8] in 1986. After this ground breaking work, the theory of frames began to be widely studied. Frames are redundant systems which are useful in applied mathematics. In the theoretical direction, powerful tools from the operator theory and Banach spaces are being employed to study frames. An excellent approach towards the utility of frames in different directions can be found in so nice books [3, 5] and in the paper by Heil and Walnut [13].

Let $\mathcal{H}$ be a real (or complex) separable Hilbert space. A countable system $\{f_k\} \subset \mathcal{H}$ is called a *frame* (or *Hilbert frame*) for $\mathcal{H}$, if there exist finite positive constants A and B such that

$$A\|f\|^2 \leq \|\{\langle f, f_k\rangle\}\|_{\ell^2}^2 \leq B\|f\|^2 \text{ for all } f \in \mathcal{H}. \tag{1}$$

The positive constants A and B are called *lower* and *upper frame bounds* of the frame, respectively. They are not unique. The inequality in (1) is called the *frame inequality* of the frame. The operator $U : \ell^2 \to \mathcal{H}$ defined as

$$U(\{c_k\}) = \sum_{k=1}^{\infty} c_k f_k, \ \{c_k\} \in \ell^2$$

is called the *pre-frame operator* or the *synthesis operator* and its adjoint operator $U^* : \mathcal{H} \to \ell^2$ given by

$$U^*(f) = \{\langle f, f_k\rangle\} \text{ for all } f \in \mathcal{H}$$

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is called the *analysis operator*.

Composing U and U^* we obtain the *frame operator* $S = UU^* : \mathcal{H} \to \mathcal{H}$ which is given by

$$S(f) = \sum_{k=1}^{\infty} \langle f, f_k \rangle f_k \text{ for all } f \in \mathcal{H}.$$

The frame operator S is a positive, self-adjoint and invertible operator on $\mathcal{H}$. For all $f \in \mathcal{H}$, we have

$$f = SS^{-1}f = \sum_{k=1}^{\infty} \langle S^{-1}f, f_k \rangle f_k = \sum_{k=1}^{\infty} \langle f, S^{-1}f_k \rangle f_k.$$

The series converges unconditionally and is called the *reconstruction formula* for the frame. The representation of f in the reconstruction formula need not be unique. Gröchenig, generalized Hilbert frames to Banach spaces in [12]. Before the concept of Banach frames was formalized, it appeared in the fundamental work of Feichtinger and Gröchenig [9, 10] related to the *atomic decompositions*. Atomic decompositions appeared in the field of applied mathematics providing many applications. An atomic decomposition allow a representation of every vector of the space via a series expansion in terms of a fixed sequence of vectors which we call *atoms*. On the other hand Banach frame for a Banach space ensure reconstruction via a bounded linear operator or *synthesis operator*. During the development of frames and expansions systems (redundant building blocks), in the later half of twentieth century, Coifman and Weiss in [6] introduced the notion of atomic decomposition for function spaces. This concept was further generalized by Gröchenig [12], who introduced the notion of Banach frames for Banach spaces. Casazza, Han and Larson discussed a deep analysis on frames and atomic decompositions and Banach frames in [4].

Definition 1.1. [12] *Let $\mathcal{X}$ be an infinite dimensional Banach space over the field $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), $\mathcal{X}^*$ its conjugate space and let $\mathcal{X}_d$ be an associated Banach space of scalar valued sequences indexed by $\mathbb{N}$. Let $\{f_k^*\} \subset \mathcal{X}^*$ and $\mathcal{S} : \mathcal{X}_d \to \mathcal{X}$ be a given operator. The pair $\mathcal{F} \equiv (\{f_k^*\}, \mathcal{S})$ is called a* Banach frame *for $\mathcal{X}$ with respect to $\mathcal{X}_d$ if*

(1) *$\{f_k^*(f)\} \in \mathcal{X}_d$ for all $f \in \mathcal{X}$,*

(2) *There exist positive constants A and B with $0 < A \leq B < \infty$ such that*

$$A\|f\|_{\mathcal{X}} \leq \|\{f_k^*(f)\}\|_{\mathcal{X}_d} \leq B\|f\|_{\mathcal{X}} \text{ for all } f \in \mathcal{X}, \tag{2}$$

(3) *$\mathcal{S}$ is a bounded linear operator such that*

$$\mathcal{S}(\{f_k^*(f)\}) = f \text{ for all } f \in \mathcal{X}\,.$$

The positive constants A and B are called *lower* and *upper frame bounds* of the Banach frame $(\{f_n^*\}, \mathcal{S})$, respectively. The operator $\mathcal{S} : \mathcal{X}_d \to \mathcal{X}$ is called the *reconstruction operator* (or the *pre-frame operator*) and the inequality (2) is called the *Banach frame inequality.* The Banach frame $(\{f_k^*\}, \mathcal{S})$ is said to be *tight* if $A = B$ and *normalized tight* if $A = B = 1$. If there is no reconstruction operator $\mathcal{S}_j$ ($j \in \mathbb{N}$) such that $(\{f_k\}_{k \neq j}, \mathcal{S}_j)$ is a Banach frame for $\mathcal{X}^*$, then $(\{f_k^*\}, \mathcal{S})$ is called an *exact Banach frame* for $\mathcal{X}$. For a type of a retro Banach frame associated with a boundary value problem, see [17].

An operator $T : X \to Y$ from a normed space X into a normed space Y is said to be *compact linear operator* if T is linear and if for every bounded subset $M \subset X$, the image $T(M)$ is relatively compact.

Motivation: Aldroubi in [1] introduced two methods for generating frames of a Hilbert space $\mathcal{H}$. In one of his method, one approach is to construct frames for $\mathcal{H}$ which are images of a given frame for $\mathcal{H}$ under a bounded linear operator on $\mathcal{H}$. The other method uses bounded linear operator on ℓ^2 to generate frames of $\mathcal{H}$. In this paper, we discuss the construction of a Banach frame from a compact linear operator on the conjugate space of the underlying space. Let $\mathcal{F} \equiv (\{f_k^*\}, \mathcal{S})$ be a Banach frame for $\mathcal{X}$ and let Θ be a compact linear operator on $\mathcal{X}^*$. Then, in general, there exists no reconstruction operator $\mathcal{S}_0$ such that $(\{\Theta(f_k^*)\}, \mathcal{S}_0)$ is a Banach frame for $\mathcal{X}$. This is justified in the following example.

Example. Let $\mathcal{X} = L^2(\Omega, \mu)$ be the discrete signal space, where $\Omega = \mathbb{N}$ and μ is the counting measure. Let $\{\chi_k\} \subset \mathcal{X}$ be a total orthonormal system, where $\chi_k = \{0, 0, .., \underbrace{1}_{kth}, 0, 0, ...\}$ $(k \in \mathbb{N})$ and let $\{\chi_k^*\}$ be the corresponding dual system for $\{\chi_k\}$, i.e., $\chi_k^*(\chi_l) = \delta_{k,l}$ for all $k, l \in \mathbb{N}$. Then, there exists a reconstruction operator $\mathcal{S}$ such that $(\{\chi_k^*\}, \mathcal{S})$ is a Banach frame (normalized tight) for $\mathcal{X}$ with respect to $\mathcal{Z}_d = \mathcal{X}$. Let Θ_0 be the weighted shift operator on $\mathcal{X}^*$ which is represented by the infinite matrix

$$\Theta_0 \leftrightarrow \begin{bmatrix} 0 & 0 & 0 & .. & 0 & ...... \\ 0 & 1 & 0 & .. & 0 & ...... \\ 0 & 0 & \frac{1}{4} & 0 & .. & ...... \\ 0 & 0 & 0 & \frac{1}{9} & .. & ...... \\ .. & .. & .. & .. & .. & ...... \\ .. & .. & .. & .. & .. & ...... \\ .. & .. & .. & .. & .. & ...... \end{bmatrix}.$$

Then, it is easy to verify that Θ_0 is a compact linear operator on $\mathcal{X}^*$. Furthermore, there exists no reconstruction operator Ξ_0 such that $(\Theta_0(\chi_k^*)\}, \Xi_0)$ is a Banach frame for $\mathcal{X}$ with respect to some associated Banach space $\mathcal{Z}_d$. Indeed, let $(A_0,\ B_0)$ be a choice of frame bounds for $(\{\Theta_0(\chi_k^*)\}, \Xi_0)$.
Then

$$A_0\|f\| \leq \|\Theta_0(\chi_k^*)(f)\}\|_{\mathcal{Z}_d} \leq B_0\|f\| \text{ for all } f \in \mathcal{X}. \tag{3}$$

Choose $f_0 = \chi_1$. Then, we have $\Theta_0(\chi_k^*)(f_0) = 0$ for all $k \in \mathbb{N}$. Therefore, by using the Banach frame inequality (3), we obtain $f_0 = 0$, a contradiction. Thus, the image of a Banach frame (even exact) under a compact linear operator on $\mathcal{X}^*$ (associated with the orthonormal system for $\mathcal{X}$) need not be a Banach frame for the underlying space.

It is difficult to characterize compact linear operators which can generate a Banach frame for the underlying space. In this paper, we discuss a method for the construction of a compact linear operator associated with the orthonormal system of a boundary value problem which can generate a Banach frame for the underlying space.

2. The main result

Let $\mathcal{X} = L^2(a, b)$. Consider a boundary value problem(BVP) with a set of n boundary conditions.

$$(\maltese) \qquad \textbf{BVP} \quad \equiv \quad \nabla(f) = \lambda f, \quad \Lambda(f) = 0,$$

where $\nabla(\bullet) = (\bullet)^n + \Phi_1(\xi)(\bullet)^{n-1} + ... + \Phi_n(\xi)(\bullet)$ is a linear differential operator with $\Phi_j \in C^{n-k}[a, b]$, and $\Lambda(f) = 0$ denotes the set of n boundary conditions given by

$$\Lambda_j(\Phi) = \Sigma_{k=1}^n[\alpha_{j,k}\Phi^{k-1}(a) + \beta_{j,k}\Phi^{k-1}(b)] = 0,\ 1 \leq j \leq n.$$

The BVP (✠) admits a system $\{\Phi_n(\xi)\}$ and $\{\Psi_n(\xi)\}$ consisting of eigenfunction associated with the BVP (✠) (see [14] p. 66) such that

$$\Phi_n(\xi) = A_n \left[cos\frac{2\pi n\xi}{b-a} + O(\frac{1}{n})\right] \text{ and } \Psi_m(\xi) = B_m \left[sin\frac{2\pi m\xi}{b-a} + O(\frac{1}{m})\right],$$

where $n, m \in \mathbb{N} \cup \{0\}$.
Note that

$$\left\|\Phi_n(\xi) - A_n cos\frac{2\pi nt}{b-a}\right\|^2 < 1 \text{ and } \left\|\Psi_m(\xi) - B_m sin\frac{2\pi mt}{b-a}\right\|^2 < 1.$$

Let us write $\{f_k\} = \{\Phi_n\} \bigsqcup \{\Psi_m\}$. By using Paley and Weiner Theorem [16, p. 208], $\{f_k\}$ form a total orthonormal system for $\mathcal{X}$. Let $\{f_k^*\}$ be the corresponding dual system in $\mathcal{X}^*$. That is, $f_i^*(f_j) = \delta_{i,j}$ for all $i, j \in \mathbb{N}$. Let $\mathcal{Z}_d = \{\{f_k^*(f)\} : f \in \mathcal{X}\}$. Then, $\mathcal{Z}_d$ is a Banach space with the norm given by

$$\|\{f_k^*(f)\}\|_{\mathcal{Z}_d} = \|f\|_{\mathcal{X}}, f \in \mathcal{X}. \tag{4}$$

Define $\mathcal{S} : \mathcal{Z}_d \to \mathcal{X}$ by $S(\{f_k^*(f)\}) = f$, $f \in \mathcal{X}$. Then, $\mathcal{S}$ is a bounded linear operator such that $\mathcal{F}_0 \equiv (\{f_k^*\}, \mathcal{S})$ is a Banach frame for $\mathcal{X}$ with respect to $\mathcal{Z}_d$ and with bounds $A = B = 1$. The pair $\mathcal{F}_0$ is called the Banach frame for $\mathcal{X}$ associated with the BVP (✠). Define $\Theta : \mathcal{X}^* \to \mathcal{X}^*$ by

$$\Theta(f^*) = \sum_{n=1}^{\infty} \frac{1}{n^2}\langle f^*, f_n^*\rangle f_n^*. \tag{5}$$

Then, Θ is a compact linear operator on $\mathcal{X}^*$. Next, we show that there exists a reconstruction operator $\mathcal{S}_0$ such that $(\{\Theta(f_k^*)\}, \mathcal{S}_0)$ is a Banach frame for $\mathcal{X}$. On the contrary, assume that there exists no reconstruction operator $\mathcal{S}_0$ such that $(\{\Theta(f_k^*)\}, \mathcal{S}_0)$ is a Banach frame for $\mathcal{X}$ with respect to any associated Banach space of scalar valued sequences. Then, by using the Hahn-Banach theorem there exists a nonzero vector $f_0 \in \mathcal{X}$ such that

$$(\Theta f_k^*)(f_0) = 0 \text{ for all } k \in \mathbb{N}.$$

We compute

$$\begin{aligned} 0 &= (\Theta f_k^*)(f_0) \\ &= \left(\sum_{n=1}^{\infty} \frac{1}{n^2}\langle f_k^*, f_n^*\rangle f_n^*\right)(f_0) \\ &= \left(\frac{1}{k^2} f_k^*\right)(f_0) \text{ for all } k \in \mathbb{N}. \end{aligned}$$

This gives $f_k^*(f_0) = 0$ for all $k \in \mathbb{N}$. Thus, by using (4), we have $f_0 = 0$, a contradiction. Hence we can find a reconstruction operator $\mathcal{S}_0$ such that $(\{\Theta(f_k^*)\}, \mathcal{S}_0)$ is a Banach frame for $\mathcal{X}$ with respect to an associated Banach space of scalar valued sequences. This is summarized in the following theorem.

Theorem 2.1. *Let $\mathcal{X} = L^2(a, b)$ and let $(\{f_k^*\}, \mathcal{S})$ be the Banach frame for $\mathcal{X}$ associated with the BVP (✠) . Then, there exists a compact linear operator $\Theta : \mathcal{X}^* \to \mathcal{X}^*$ and a reconstruction operator $\mathcal{S}_0$ such that $(\{\Theta(f_k^*)\}, \mathcal{S}_0)$ is a Banach frame for $\mathcal{X}$.*

3. Concluding remark

The compact linear operator associated with the orthonormal system in a Hilbert space, in general, fails to generate a Banach frame for the underlying space (see Example 1.1). The compact linear operator given in (5) (and generalized for its prototype) can generate a Banach frame for the underlying space and is called the compact linear operator associated with the BVP (✠). This can be generalized to an arbitrary Hilbert space with a total orthonormal system.

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